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On the Root and Integral Closure of Noetherian Domains of Dimension One

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I. INTRODUCTION

Let R be an integral domain with quotient field K. For an integer $n \geqslant 1$, R is said to be *n-root closed* if whenever $x \in K$ with $x^n \in R$, then $x \in R$; R is called *root closed*, if R is n-root closed for all integers $n > 1$. Obviously, any integrally closed domain is root closed. The converse is false, as is shown in [3; Ex. 6, p. 184]; other examples are contained in Section III. More precisely, we prove in Section III that if d is any square-free integer, then $d \equiv 1 \pmod 8$ iff $\mathbb{Z}[\sqrt{d}]$ is root closed and not integrally closed. Nevertheless, we shall show in this note that there are many situations in which a root closed Noetherian domain of dimension one is automatically integrally closed. This is the case, for example, if R is any subring of a number field L (i.e., $[L : \mathbb{Q}] < \infty$) with $\frac{1}{2} \in R$, or R is the coordinate ring of an irreducible affine curve over a finite field with at least three elements. If R is the coordinate ring of an irreducible affine curve over an algebraically closed field k, even more is true: R is integrally closed if R is n-root closed for some integer $n > 1$ prime to the characteristic of k. This sharpens [2; Theorem 3]. We note that if R is n-root closed for some integer $n > 1$, R is (2, 3)-closed in the sense of [2]: for let $x \in K$ with $x^2, x^3 \in R$; then $x^m \in R$ for all $m \geqslant 2$, i.e., $x^n \in R$ and thus $x \in R$. It seems that this triviality has been overlooked in [2]. Moreover, if R is the coordinate ring of an irreducible affine curve over a real closed field and R is 2-root closed, then R is integrally closed. In particular, if R is the coordinate ring of an irreducible affine curve over the field $\mathbb{R}$ or $\mathbb{C}$, and R is 2-root closed, then R is integrally closed.

In Section II we isolate the key point of this note in the Main Lemma. This is applied in Sections III and IV to the situations mentioned above.

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II. The Key Point

Lemma 1 is needed in the proof of the Main Lemma.

Lemma 1. *Let R be an integral domain and S any subring of the quotient field of R such that $R \subseteq S$. If R is n-root closed for some integer $n > 1$, the conductor $R : S$ of R in S is a radical ideal in S.*

Proof. Let $s \in S$ such that $s^m \in R : S$ for some integer $m \geqslant 1$. If $m > 1$, consider the smallest integer $l \geqslant 1$ with $ln \geqslant m$. As $m > 1$ and $n > 1$, one has $l < m$. Now let $t \in S$. Then $(s^l t)^n = s^{ln} t^n \in R$, whence $s^l t \in R$. This proves $s^l \in R : S$. An easy induction argument now shows that $s \in R : S$ and this finishes the proof.

Main Lemma. *Let $(R, \mathfrak{m})$ be a local Noetherian domain of dimension one with finite integral closure S. Further, let $n > 1$ be an integer. If R is n-root closed and for some maximal ideal M of S, S/M contains a primitive n-th root of unity, then S is a local ring with maximal ideal $\mathfrak{m}$.*

Proof. We observe that S is a semi-local Dedekind domain [1; VII, Sect. 2.5, Proposition 5]. Consider the conductor $R : S$ of R in S. If $R : S = R$, then $R = S$ and all is proven. So assume $R : S \neq R$, i.e., $R : S \subseteq \mathfrak{m}$. As S is a finite R-module, $R : S \neq 0$. Therefore, by Lemma 1, $R : S$ is the intersection of maximal ideals of S. But this implies $R : S = \mathfrak{m} = M_1 \cap \cdots \cap M_r$, where M_i, $i = 1, \ldots, r$, are the maximal ideals of S. So it is sufficient to show $r = 1$.

Assume $r > 1$ and choose an element $\zeta \in S$ and an integer i $(1 \leqslant i \leqslant r)$ such that ζ is a primitive nth root of unity $\bmod M_i$. By the Chinese remainder theorem [1; II, Sect. 1.2, Proposition 5] there is an element $s \in S$ such that $s - \zeta \in M_i$ and $s - 1 \in M_j$ for $j \neq i$, $1 \leqslant j \leqslant r$. Write $s = \zeta + x = 1 + y$, $\zeta^n = 1 + z$, where $x, z \in M_i$, $y \in \bigcap_{j \neq i} M_j$. Then

$$\begin{aligned} s^n &= \zeta^n + \sum_{i=1}^{n} \binom{n}{i} \zeta^{n-i} x^i = 1 + z + \sum_{i=1}^{n} \binom{n}{i} \zeta^{n-i} x^i \\ &= 1 + \sum_{i=1}^{n} \binom{n}{i} y^i. \end{aligned}$$

This yields $z + \sum_{i=1}^{n} \binom{n}{i} \zeta^{n-i} x^i = \sum_{i=1}^{n} \binom{n}{i} y^i \in M_i \cap \bigcap_{j \neq i} M_j = \mathfrak{m}$. In particular, $s^n \in R$ and thus $s \in R$. This implies $y = s - 1 \in R \cap M_j = \mathfrak{m}$ for $j \neq i$, $1 \leqslant j \leqslant r$ and so $\zeta - 1 = y - x \in M_i$ in contradiction to the primitivity of $\zeta \bmod M_i$. So $r = 1$ and the proof is finished.

To obtain global results, we shall use Lemma 2, which is included here only for the sake of completeness.

LEMMA 2. *Let R be an integral domain and $n > 1$ be an integer. The following assertions are equivalent*:

(1) *R is n-root closed (resp. root closed, resp. integrally closed).*

(2) *For every maximal ideal $\mathfrak{m}$ of R, the local ring $R_{\mathfrak{m}}$ is n-root closed (resp. root closed, resp. integrally closed).*

Proof. Straightforward.

III. ARITHMETICAL SITUATIONS

THEOREM 1. *Let R be a Noetherian domain of dimension one such that the integral closure S of R is a finite R-module and for every maximal ideal M of S, S/M is a finite field with at least 3 elements. If R is root closed, then R is integrally closed.*

Proof. By Lemma 3, we can assume that R is a local ring; let $\mathfrak{m}$ denote the maximal ideal of R. Choose any maximal ideal M of S. Then S/M is a finite field, whence contains a primitive nth root of unity, where $n := \operatorname{card}(S/M) - 1 > 1$. By the Main Lemma, S is a local ring with maximal ideal $\mathfrak{m}$. Choose $\zeta \in S$ such that ζ is a primitive nth root of unity mod $\mathfrak{m}$. Then $\zeta^n - 1 \in \mathfrak{m} \subseteq R$, i.e., $\zeta^n \in R$, whence $\zeta \in R$. Thus $R/\mathfrak{m} = S/\mathfrak{m}$ and so $R = S$, that is, R is integrally closed.

As is well known, if R is any subring of a number field L (i.e., L is a finite extension of $\mathbb{Q}$), R is a Noetherian domain such that the integral closure S of R is a finite R-module and for every maximal ideal M of S, S/M is a finite field [1; VII, Sect. 2.5, Propositions 5 and 3, 40.3, 41.8].

COROLLARY 1. *Let R be any subring of a number field such that for every maximal ideal M of the integral closure S of R with $2 \in M$, S/M has more than 2 elements.*

If R is root closed, then R is integrally closed.

Proof. The condition on the prime number 2 says that for every maximal ideal M of S such that $\operatorname{char}(S/M) = 2$, S/M has at least 4 elements. As any field of characteristic $p \geqslant 3$ has at least 3 elements, the assertion follows from Theorem 1.

A special case of Corollary 1 is

COROLLARY 2. *Let R be any subring of a number field such that $\frac{1}{2} \in R$. If R is root closed, then R is integrally closed.*

The following proposition shows that the corollaries become false, if the assumption on the number 2 is omitted.

PROPOSITION. *Let d be a square-free rational integer. Then* $d \equiv 1 \pmod 8$ *iff* $\mathbb{Z}[\sqrt{d}]$ *is root closed and not integrally closed.*

Proof. As is well known, $\mathbb{Z}[\sqrt{d}]$ is not integrally closed iff $d \equiv 1 \pmod 4$. So we have to show that $\mathbb{Z}[\sqrt{d}]$ is root closed for $d \equiv 1 \pmod 8$ and not root closed for $d \equiv 5 \pmod 8$. Let $w = \frac{1}{2}(1 + \sqrt{d})$. Then $\mathbb{Z}[w]$ is the integral closure of $\mathbb{Z}[\sqrt{d}]$, if $d \equiv 1 \pmod 4$. Let $d \equiv 1 \pmod 8$. It is sufficient to show that if $x \in \mathbb{Z}[w] \backslash \mathbb{Z}[\sqrt{d}]$, then $x^n \in \mathbb{Z}[w] \backslash \mathbb{Z}[\sqrt{d}]$ for all integers $n \geqslant 1$. Let $x = u + vw \in \mathbb{Z}[w] \backslash \mathbb{Z}[\sqrt{d}]$, where $u, v \in \mathbb{Z}$; $x \notin \mathbb{Z}[\sqrt{d}]$ means $v \equiv 1 \pmod 2$. We prove now by induction that for all integers $n \geqslant 1$,

$$x^n = r + sw, \qquad (*)$$

where $r, s \in \mathbb{Z}$, $r \equiv u \pmod 2$, $s \equiv 1 \pmod 2$, showing thus $x^n \notin \mathbb{Z}[\sqrt{d}]$.

The case $n = 1$ is clear and so assume $(*)$ to be true for an integer $n \geqslant 1$. Observe that $w^2 = w - l$ with $l = (1-d)/4 \equiv 0 \pmod 2$. So multiplying $(*)$ with x yields

$$x^{n+1} = (r + sw)(u + vw) = (ur - svl) + (rv + us + sv)w$$

with $ur - svl \equiv u \pmod 2$ and $rv + us + sv \equiv 1 \pmod 2$. This proves that $\mathbb{Z}[\sqrt{d}]$ is root closed, if $d \equiv 1 \pmod 8$. Let $d \equiv 5 \pmod 8$. Then $2\mathbb{Z}[w]$ is a prime ideal of $\mathbb{Z}[w]$, and so, by Corollary 1, $\mathbb{Z}[\sqrt{d}]$ cannot be root closed. Alternatively, this can be proven by observing that $w^2 = w - l$, where $l = \frac{1}{4}(1-d) \equiv 1 \pmod 2$ and thus

$$w^3 = (w - l)w = (1 - l)w - l \in \mathbb{Z}[\sqrt{d}].$$

Corollary 3 of Theorem 1 leads over to the geometrical situations considered in Section IV. We note that if R is the coordinate ring of an irreducible affine variety over any field, then the integral closure of R is a finite R-module [1; V, Sect. 3.2, Theorem 2].

COROLLARY 3. *Let R be the coordinate ring of an irreducible affine algebraic curve over a finite field k with at least 3 elements. If A is root closed, A is integrally closed.*

Proof. If M is any maximal ideal of the integral closure S of R, then S/M is a finite extension field of k. So the theorem applies.

IV. Geometrical Situations

THEOREM 2. *Let R be the coordinate ring of an irreducible affine algebraic curve over an algebraically closed field k of characteristic $p \geqslant 0$. Further, let $n > 1$ be an integer such that p does not divide n.*

If R is n-root closed, then R is integrally closed.

Proof. By Lemma 2, we have to show that for every maximal ideal $\mathfrak{m}$ of R, $R_{\mathfrak{m}}$ is integrally closed. As p does not divide n, k contains a primitive nth root of unity. So the Main Lemma implies that the integral closure S of $R_{\mathfrak{m}}$ is a local ring with maximal ideal $\mathfrak{m}R_{\mathfrak{m}}$; but $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} = k = S/\mathfrak{m}R_{\mathfrak{m}}$ and so $R_{\mathfrak{m}} = S$.

COROLLARY. *Let R be the coordinate ring of an irreducible affine algebraic curve over an algebraically closed field of characteristic $p \geqslant 0$.*

If $p \neq 2$ and R is 2-root closed, then R is integrally closed. If $p = 2$ and R is 3-root closed, then R is integrally closed.

One has a similar result for real closed fields.

THEOREM 3. *Let R be the coordinate ring of an irreducible affine algebraic curve over a real closed field k. If R is 2-root closed, then R is integrally closed.*

Proof. By Lemma 2 and the Main Lemma, for every maximal ideal $\mathfrak{m}$ of R, the integral closure S of $R_{\mathfrak{m}}$ is a local ring with maximal ideal $\mathfrak{m}R_{\mathfrak{m}}$. But $k \subseteq R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} \subseteq S/\mathfrak{m}R_{\mathfrak{m}} \subseteq k(\sqrt{-1})$, because k is a real closed field. If $S/\mathfrak{m}R_{\mathfrak{m}} = k(\sqrt{-1})$, choose $\zeta \in S$ such that $\zeta^2 + 1 \in \mathfrak{m}R_{\mathfrak{m}}$; then $\zeta^2 \in R_{\mathfrak{m}}$, whence $\zeta \in R_{\mathfrak{m}}$. So, in any case, $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} = S/\mathfrak{m}R_{\mathfrak{m}}$, and so $R_{\mathfrak{m}} = S$.

In particular, if R is the coordinate ring of an irreducible affine algebraic curve over $\mathbb{R}$ or $\mathbb{C}$, and R is 2-root closed, then R is integrally closed.

References

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